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Integrable *XYZ* model with staggered anisotropy parameter

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Abstract
We apply to the *XYZ* model the technique of construction of integrable models with staggered parameters, presented recently for the *XXZ* case by Arnaudon *et al*. The solution of modified Yang–Baxter equations is found and the corresponding integrable zig-zag ladder Hamiltonian is calculated. The result coincides with the *XXZ* case in the appropriate limit.

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1. Introduction

Recently a technique was proposed [1] which allows us to construct a zig-zag ladder type of integrable model on the basis of known integrable chain models. It was successfully realized on models with basic $sl(n)$ symmetries: the $sl(2)$ case (*XXZ* model) in [1], the $sl(3)$ case (anisotropic t–J model) in [2] and the general $sl(n)$ case in [3]. The main element of this construction is the possibility to stagger the parameters of the known integrable R-matrix along chain and time directions. Together with alternation of the anisotropy parameter Δ [1, 2], the staggered shift of the spectral parameter u by some additional model parameter θ was also introduced, which caused the appearance of the next-to-nearest-neighbour interaction terms in the resulting local Hamiltonian.

The motivation for considering this type of integrable model is based on the observation that the phenomenological Chalker–Coddington model [4] for edge excitations in the Hall effect after its reformulation as a lattice field theory [5] and calculation of disorder over random phases becomes a Hubbard-type model with staggered disposition of R-matrices [6]. Therefore it is meaningful to start the investigation of staggered models from simple cases.

In this paper we generalize the construction of the staggered *XXZ* model and construct a staggered integrable *XYZ* model. The corresponding generalized Yang–Baxter equations (YBEs), which are the condition of integrability [7, 8], have a solution. Therefore, one can define a model on a two-leg ladder, the Hamiltonian of which is presented here.

2. Basic definitions and staggered YBEs

We start by writing down the modification of basic ingredients of the algebraic Bethe ansatz (ABA) technique [7, 8] appropriate for our purposes.

Let us consider $\mathbb{Z}_2$ graded quantum $V_{j,\rho}(v)$ (with $j = 1, \dots, N$ as a chain index) and auxiliary $V_{a,\sigma}(u)$ spaces, where $\rho, \sigma = 0, 1$ are the grading indices. Consider R-matrices, which act on the direct product of spaces $V_{a,\sigma}(u)$ and $V_{j,\rho}(v)$, $(\sigma, \rho = 0, 1)$, mapping them on the intertwined direct product of $V_{a,\bar{\sigma}}(u)$ and $V_{j,\bar{\rho}(v)}$ with the complementary $\bar{\sigma} = (1-\sigma)$, $\bar{\rho} = (1-\rho)$ indices

$$R_{aj,\sigma\rho}(u, v) \; : \; V_{a,\sigma}(u) \otimes V_{j,\rho}(v) \to V_{j,\bar{\rho}}(v) \otimes V_{a,\bar{\sigma}}(u). \tag{2.1}$$

It is convenient to introduce two transmutation operations ι_1 and ι_2 with the property $\iota_1^2 = \iota_2^2 = \mathrm{id}$ for the quantum and auxiliary spaces correspondingly, and to define the operators $R_{aj,\sigma\rho}$ as follows:

$$\begin{aligned} R_{aj,00} &\equiv R_{aj}, \qquad & R_{aj,01} &\equiv R_{aj}^{\iota_1}, \\ R_{aj,10} &\equiv R_{aj}^{\iota_2}, \qquad & R_{aj,11} &\equiv R_{aj}^{\iota_1\iota_2}. \end{aligned} \tag{2.2}$$

The introduction of the $\mathbb{Z}_2$ grading of quantum spaces in the time direction means that we now have two monodromy operators T_ρ, $\rho = 0, 1$, which act on the space $V_\rho(u) = \prod_{j=1}^N V_{j,\rho}(u)$ by mapping it on $V_{\bar{\rho}}(u) = \prod_{j=1}^N V_{j,\bar{\rho}}(u)$

$$T_\rho(v, u) : V_\rho(u) \to V_{\bar{\rho}}(u), \qquad \rho = 0, 1. \tag{2.3}$$

It is now clear that the monodromy operator of the model, which is defined by translational invariance by two steps in the time direction and which determines the partition function, is the product of two monodromy operators

$$T(v, u) = T_0(v, u) T_1(v, u). \tag{2.4}$$

The $\mathbb{Z}_2$ grading of auxiliary spaces along the chain direction means that the $T_0(u, v)$ and $T_1(u, v)$ monodromy matrices are defined according to the following staggered product of the $R_{aj}(v, u)$ and $\bar{R}_{aj}^{\iota_2}(v, u)$ matrices:

$$\begin{aligned} T_1(v, u) &= \prod_{j=1}^N R_{a,2j-1}(v, u) \bar{R}_{a,2j}^{\iota_2}(v, u), \\ T_0(v, u) &= \prod_{j=1}^N \bar{R}_{a,2j-1}^{\iota_1}(v, u) R_{a,2j}^{\iota_1\iota_2}(v, u), \end{aligned} \tag{2.5}$$

where the notation $\bar{R}$ denotes a different parametrization of the $R(v, u)$-matrix via spectral parameters v and u and can be considered as an operation over R with property $\bar{\bar{R}} = R$. For the integrable models where the intertwiner matrix $R(v-u)$ simply depends on the difference of the spectral parameters v and u this operation is a shift of its argument u:

$$\bar{R}(u) = R(\bar{u}), \qquad \bar{u} = \theta - u, \tag{2.6}$$

where θ is an additional model parameter. We shall consider this case in this paper.

As is well known in Bethe ansatz technique [7, 8], the sufficient condition for the commutativity of transfer matrices $\tau(u) = \mathrm{Tr}\, T(u)$ with different spectral parameters is the YBE. For our case we have a set of two equations [1]

$$R_{12}(u,v)\bar{R}^{\iota_1}_{13}(u)R_{23}(v) = R^{\iota_1}_{23}(v)\bar{R}_{13}(u)\tilde{R}_{12}(u,v) \tag{2.7}$$

and

$$\tilde{R}_{12}(u,v)R^{\iota_1\iota_2}_{13}(u)\bar{R}^{\iota_2}_{23}(v) = \bar{R}^{\iota_1\iota_2}_{23}(v)R^{\iota_2}_{13}(u)R_{12}(u,v). \tag{2.8}$$

Indeed, the integrability is proved if two two-row transfer matrices with different spectral parameters u and v commute. To ensure this with the usual railway argument, we need to insert four intertwiner R-matrices (and their inverses): two ordinary ones to exchange occurrences of $R^{\iota_1}(u)R(v)$ and two $\tilde{R}(u,v)$ to intertwine $R(u)R^{\iota_1}(v)$. The use of the set of YBEs (2.7) and (2.8) precisely performs the needed exchanges.

3. Staggered *XYZ* Heisenberg chain

The R-matrix of the ordinary *XYZ* Heisenberg model can be represented as

$$R(u) = \begin{pmatrix} a(u) & & & d(u) \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ d(u) & & & a(u) \end{pmatrix}. \tag{3.1}$$

Inputting this expression of $R(u)$ into the YBE (2.7) we obtain the following three sets of equations on $a(u)$, $b(u)$, $c(u)$, $d(u)$. The first set is

$$\begin{aligned}
&a(u,v)a^{\iota_1}(\bar{u})a(v) + d(u,v)c^{\iota_1}(\bar{u})d(v) = a^{\iota_1}(v)a(\bar{u})\tilde{a}(u,v) + d^{\iota_1}(v)c(\bar{u})\tilde{d}(u,v),\\
&a(u,v)b^{\iota_1}(\bar{u})b(v) + d(u,v)d^{\iota_1}(\bar{u})c(v) = b^{\iota_1}(v)b(\bar{u})\tilde{a}(u,v) + c^{\iota_1}(v)d(\bar{u})\tilde{d}(u,v),\\
&b(u,v)b^{\iota_1}(\bar{u})a(v) + c(u,v)d^{\iota_1}(\bar{u})d(v) = a^{\iota_1}(v)b(\bar{u})\tilde{b}(u,v) + d^{\iota_1}(v)d(\bar{u})\tilde{c}(u,v),\\
&b(u,v)a^{\iota_1}(\bar{u})b(v) + c(u,v)c^{\iota_1}(\bar{u})c(v) = b^{\iota_1}(v)a(\bar{u})\tilde{b}(u,v) + c^{\iota_1}(v)c(\bar{u})\tilde{c}(u,v),
\end{aligned} \tag{3.2}$$

which are satisfied automatically in case of trivial ι_1 operation, as happened in the ordinary *XYZ* model.

The second set

$$\begin{aligned}
&a(u,v)b^{\iota_1}(\bar{u})c(v) + d(u,v)d^{\iota_1}(\bar{u})b(v) = c^{\iota_1}(v)a(\bar{u})\tilde{b}(u,v) + b^{\iota_1}(v)c(\bar{u})\tilde{c}(u,v),\\
&b(u,v)a^{\iota_1}(\bar{u})c(v) + c(u,v)c^{\iota_1}(\bar{u})b(v) = c^{\iota_1}(v)b(\bar{u})\tilde{a}(u,v) + b^{\iota_1}(v)d(\bar{u})\tilde{d}(u,v),\\
&a(u,v)c^{\iota_1}(\bar{u})a(v) + d(u,v)a^{\iota_1}(\bar{u})d(v) = c^{\iota_1}(v)a(\bar{u})\tilde{c}(u,v) + b^{\iota_1}(v)c(\bar{u})\tilde{b}(u,v),\\
&c(u,v)a^{\iota_1}(\bar{u})c(v) + b(u,v)c^{\iota_1}(\bar{u})b(v) = a^{\iota_1}(v)c(\bar{u})\tilde{a}(u,v) + d^{\iota_1}(v)a(\bar{u})\tilde{d}(u,v),\\
&b(u,v)c^{\iota_1}(\bar{u})c(v) + c(u,v)a^{\iota_1}(\bar{u})b(v) = a^{\iota_1}(v)b(\bar{u})\tilde{c}(u,v) + d^{\iota_1}(v)d(\bar{u})\tilde{b}(u,v),\\
&b(u,v)d^{\iota_1}(\bar{u})d(v) + c(u,v)b^{\iota_1}(\bar{u})a(v) = b^{\iota_1}(v)a(\bar{u})\tilde{c}(u,v) + c^{\iota_1}(v)c(\bar{u})\tilde{b}(u,v),
\end{aligned} \tag{3.3}$$

reduces to staggered *XXZ* model equations [1] in the case of $d(u) = 0$. Finally we have a third set of equations

$$\begin{aligned}
&a(u,v)a^{\iota_1}(\bar{u})d(v) + d(u,v)c^{\iota_1}(\bar{u})a(v) = a^{\iota_1}(v)d(\bar{u})\tilde{c}(u,v) + d^{\iota_1}(v)b(\bar{u})\tilde{b}(u,v),\\
&b(u,v)b^{\iota_1}(\bar{u})d(v) + c(u,v)d^{\iota_1}(\bar{u})a(v) = a^{\iota_1}(v)c(\bar{u})\tilde{d}(u,v) + d^{\iota_1}(v)a(\bar{u})\tilde{a}(u,v),\\
&b(u,v)d^{\iota_1}(\bar{u})a(v) + c(u,v)b^{\iota_1}(\bar{u})d(v) = b^{\iota_1}(v)d(\bar{u})\tilde{a}(u,v) + c^{\iota_1}(v)b(\bar{u})\tilde{d}(u,v),\\
&a(u,v)d^{\iota_1}(\bar{u})b(v) + d(u,v)b^{\iota_1}(\bar{u})c(v) = a^{\iota_1}(v)d(\bar{u})\tilde{b}(u,v) + d^{\iota_1}(v)b(\bar{u})\tilde{c}(u,v),\\
&a(u,v)c^{\iota_1}(\bar{u})d(v) + d(u,v)a^{\iota_1}(\bar{u})a(v) = b^{\iota_1}(v)b(\bar{u})\tilde{d}(u,v) + c^{\iota_1}(v)d(\bar{u})\tilde{a}(u,v),\\
&a(u,v)d^{\iota_1}(\bar{u})c(v) + d(u,v)b^{\iota_1}(\bar{u})b(v) = a^{\iota_1}(v)a(\bar{u})\tilde{d}(u,v) + d^{\iota_1}(v)c(\bar{u})\tilde{a}(u,v),
\end{aligned} \tag{3.4}$$

which represents the non-trivial part of the *XYZ* model. In the *XXZ* case, when $d(u) = 0$, this set of equations disappears.

Our aim now is to find a solution of equations (3.2)–(3.4) with non-trivial ι_1 operation. It can be seen easily that by defining

$$\begin{aligned} a^{\iota_1}(u) &= a(u), & \tilde{a}(u,v) &= a(u,v), \\ b^{\iota_1}(u) &= -b(u), & \tilde{b}(u,v) &= -b(u,v), \\ c^{\iota_1}(u) &= c(u), & \tilde{c}(u,v) &= c(u,v), \\ d^{\iota_1}(u) &= -d(u), & \tilde{d}(u,v) &= -d(u,v), \end{aligned} \tag{3.5}$$

the equations (3.2) are obeyed identically, while the equations (3.3) and (3.4) reduce to the corresponding equations of the staggered case. Therefore the ordinary choice of parametrization of $a(u)$, $b(u)$, $c(u)$ and $d(u)$ via the Jacobi elliptic functions originally found by Baxter for the standard *XYZ* model [7]

$$\begin{aligned} a(u) &= \mathrm{sn}(u+\eta) = a^{\iota_1}(u), & a(u,v) &= \mathrm{sn}(\bar{u}-v+\eta) = \tilde{a}(u,v), \\ b(u) &= \mathrm{sn}u = -b^{\iota_1}(u), & b(u,v) &= \mathrm{sn}(\bar{u}-v) = -\tilde{b}(u,v), \\ c(u) &= \mathrm{sn}\eta = c^{\iota_1}(u), & c(u,v) &= \mathrm{sn}\eta = \tilde{c}(u,v), \\ d(u) &= k\,\mathrm{sn}\eta\,\mathrm{sn}u\,\mathrm{sn}(u+\eta) = -d^{\iota_1}(u), \\ d(u,v) &= k\,\mathrm{sn}\eta\,\mathrm{sn}(\bar{u}-v)\mathrm{sn}(\bar{u}-v+\eta) = -\tilde{d}(u,v), \end{aligned} \tag{3.6}$$

fulfils the remaining equations (3.3) and (3.4).

Let us now analyse the second set of YBEs (2.8). First we conclude immediately from the solution (3.5) of the previous set of YBEs that the operation $\tilde{}$ simply coincides with the operation ι_1. Then it is easy to see that if we define ι_2 operation in (2.8) as

$$R^{\iota_2}(u) = R^{\iota_1}(-u), \tag{3.7}$$

then the second set (2.8) of YBEs coincides with the first set (2.7) after additional action on it by ι_1. This means that relation (3.7) is ensuring the fulfillment of the second set (2.8) of YBEs. Therefore we have

$$\begin{aligned} a^{\iota_2}(u) &= a(-u), & b^{\iota_2}(u) &= -b(-u), \\ c^{\iota_2}(u) &= c(-u), & d^{\iota_2}(u) &= -d(-u). \end{aligned} \tag{3.8}$$

It is now necessary to emphasize that, as in the ordinary case, the parameters Δ and k defined as

$$\Delta = \frac{a^2+b^2-c^2-d^2}{2ab}, \qquad k\mathrm{sn}^2\eta = \frac{cd}{ab} \tag{3.9}$$

are constants (as well as Δ^{ι_1} and k^{ι_1}). They define the anisotropy of the model in the z- and y-directions. As one can see, now, due to solution (3.6), the anisotropy parameter Δ is staggered along the chain and time directions.

Hence, we have found the solution of staggered YBEs (2.7) and (2.8) and can now calculate the Hamiltonian of the corresponding integrable model.

4. The transfer matrix and the Hamiltonian

Having the solution of graded YBEs one can start the calculation of the monodromy matrix and the Hamiltonian. According to formula (2.4) the monodromy matrix of the model is

$$
\begin{aligned}
T^{ab,\ i_1\ldots i_{2N}}_{cd,\ k_1\ldots k_{2N}}(u,\theta) &\equiv T_0{}^{a,\ i_1\ldots i_{2N}}_{c,\ j_1\ldots j_{2N}}(u,\theta)\ T_1{}^{b,\ j_1\ldots j_{2N}}_{d,\ k_1\ldots k_{2N}}(u,\theta)\\
&= \big(R^{\iota_1}_{0,1}{}^{a\ i_1}_{a_1 j_1}(\bar u)\ R^{\iota_1\iota_2}_{0,2}{}^{a_1 i_2}_{a_2 j_2}(u)\ R^{\iota_1}_{0,3}{}^{a_2 i_3}_{a_3 j_3}(\bar u)\ldots\ R^{\iota_1\iota_2}_{0,2N}{}^{a_{2N-1} i_{2N}}_{c\quad j_{2N}}(u)\big)\\
&\quad\times\big(R_{0'1}{}^{b\ j_1}_{b_1 k_1}(u)\ R^{\iota_2}_{0',2}{}^{b_1 j_2}_{b_2 k_2}(\bar u)\ R_{0',3}{}^{b_2 j_3}_{b_3 k_3}(u)\ldots\ R^{\iota_2}_{0',2N}{}^{b_{2N-1} j_{2N}}_{d\quad k_{2N}}(\bar u)\big) \qquad (4.1)\\
&= \big(R^{\iota_1}_{0,1}{}^{a\ i_1}_{a_1 j_1}(\bar u)\ R_{0',1}{}^{b\ j_1}_{b_1 k_1}(u)\ R^{\iota_1\iota_2}_{0,2}{}^{a_1 i_2}_{a_2 j_2}(u)\ R^{\iota_2}_{0',2}{}^{b_1 j_2}_{b_2 k_2}(\bar u)\big)\ldots\\
&\quad\times\big(R^{\iota_1}_{0,2N-1}{}^{a_{2N-2} i_{2N-1}}_{a_{2N-1} j_{2N-1}}(\bar u)\ R_{0',2N-1}{}^{b_{2N-2} j_{2N-1}}_{b_{2N-1} k_{2N-1}}(u)\\
&\quad\times R^{\iota_1\iota_2}_{0,2N}{}^{a_{2N-1} i_{2N}}_{c\quad j_{2N}}(u)\ R^{\iota_2}_{0',2N}{}^{b_{2N-1} j_{2N}}_{d\quad k_{2N}}(\bar u)\big). \qquad (4.2)
\end{aligned}
$$

Let us now write an explicit formula for the R-matrix in terms of Pauli matrices

$$
\begin{aligned}
R_{0,r}{}^{ai}_{cj}(u) &= a(u)\big(\sigma_0{}^a_c\sigma_r{}^i_j + \bar\sigma_0{}^a_c\bar\sigma_r{}^i_j\big) + b(u)\big(\sigma_0{}^a_c\bar\sigma_r{}^i_j + \bar\sigma_0{}^a_c\sigma_r{}^i_j\big)\\
&\quad + c(u)\big(\sigma_0^+{}^a_c\sigma_r^-{}^i_j + \sigma_0^-{}^a_c\sigma_r^+{}^i_j\big) + d(u)\big(\sigma_0^+{}^a_c\sigma_r^+{}^i_j + \sigma_0^-{}^a_c\sigma_r^-{}^i_j\big), \qquad (4.3)
\end{aligned}
$$

where 0 and r refer to auxiliary and quantum spaces respectively and

$$
\begin{aligned}
\sigma &\equiv \tfrac{1}{2}(\sigma^4 + \sigma^z), &\qquad \bar\sigma &\equiv \tfrac{1}{2}(\sigma^4 - \sigma^z),\\
\sigma^+ &\equiv \tfrac{1}{2}(\sigma^x + \mathrm{i}\sigma^y), &\qquad \sigma^- &\equiv \tfrac{1}{2}(\sigma^x - \mathrm{i}\sigma^y).
\end{aligned}
$$

Then transfer matrix is defined as the trace of the monodromy matrix (4.1) over auxiliary spaces and at the zero value of the spectral parameter u is given by

$$
\begin{aligned}
\tau^{i_1\ldots i_{2n}}_{k_1\ldots k_{2n}}(0,\theta) &\equiv T^{ab,i_1\ldots i_{2N}}_{ab,k_1\ldots k_{2N}}(0,\theta) \qquad (4.4)\\
&= \mathrm{tr}_0\,\mathrm{tr}_{0'}\prod_{r=1}^{N}\big(R^{\iota_1}{}^{a_{2r-1} i_{2r-1}}_{a_{2r}\quad j_{2r-1}}(\theta)R^{b_{2r-1} j_{2r-1}}_{b_{2r}\quad k_{2r}}(0)R^{a_{2r}\quad i_{2r}}_{a_{2r+1} j_{2r}}(0)R^{\iota_1}{}^{b_{2r}\quad j_{2r}}_{b_{2r+1} k_{2r}}(-\theta)\big) \qquad (4.5)\\
&= \mathrm{tr}_0\,\mathrm{tr}_{0'}\prod_{r=1}^{N}\big(\mathrm{sn}^2\eta(\mathrm{sn}^2\eta - \mathrm{sn}^2\theta)P_{0,2r}P_{0',2r-1}\big) \qquad (4.6)\\
&= \big(\mathrm{sn}^2\eta(\mathrm{sn}^2\eta - \mathrm{sn}^2\theta)\big)^N\delta^{i_1}_{k_3}\delta^{i_2}_{k_4}\ldots\delta^{i_{2N-2}}_{k_{2N}}\delta^{i_{2n-1}}_{k_1}\delta^{i_{2N}}_{k_2}, \qquad (4.7)
\end{aligned}
$$

i.e. up to an overall multiplier it is becoming an operator that shifts along the chain by two units, i.e. a translation operator. Here we have used

$$
R_{0,r}{}^{ai}_{cj}(0) = \mathrm{sn}\eta\cdot\delta^i_c\delta^a_j \equiv \mathrm{sn}\eta\cdot P_{0,r}
$$

where $P_{0,r}$ is an operator permuting auxiliary (zeroth) and quantum (rth) spaces. We also used the relation (unitarity property of R)

$$
R^{\iota_1}{}^{a_{2r-1} i_{2r-1}}_{a_{2r}\quad b_{2r}}(\theta)R^{\iota_1}{}^{b_{2r} a_{2r}}_{b_{2r+1} k_{2r}}(-\theta) = \big(\mathrm{sn}^2\eta - \mathrm{sn}^2\theta\big)\,\delta^{a_{2r-1}}_{k_{2r}}\delta^{i_{2r-1}}_{b_{2r+1}},
$$

which holds due to the following identities for elliptic functions:

$$
\begin{aligned}
&a(u)a(-u) + d(u)d(-u) = \mathrm{sn}^2\eta - \mathrm{sn}^2u = b(u)b(-u) + c(u)c(-u)\\
&a(u)d(-u) + d(u)a(-u) = 0 = b(u)c(-u) + c(u)b(-u).
\end{aligned}
$$

The functions $a(u)$, $b(u)$, $c(u)$ and $d(u)$ are given by (3.6). Now we can turn to the calculation of the Hamiltonian

$$
\frac{\mathrm{d}}{\mathrm{d}u}\log\tau\bigg|_{u=0} = \mathrm{sn}^2\eta(\mathrm{sn}^2\eta - \mathrm{sn}^2\theta)\ \mathcal{H}^{j_1\ldots j_{2N}}_{k_1\ldots k_{2N}}\ P^{i_3\ldots i_{2N}\ i_1\ i_2}_{j_1\ldots j_{2N-1} j_{2N}}. \qquad (4.8)
$$

One can see that the Hamiltonian of the model is reduced to the sum of contributions from the derivatives of quartic products of neighbour R-matrices inside brackets in (4.5) after the permutation of indices. It appears that the differentiation of two terms in the bracket again

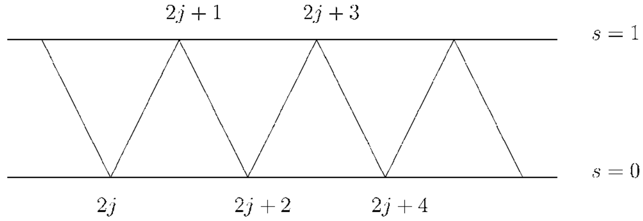

Figure 1. Zig-zag ladder chain.

contributes to the identity operator and provides a constant contribution to the Hamiltonian. Therefore the non-trivial contribution looks like

$$\begin{aligned}\big(&R^{\iota_1\ i_2i_3}_{\ \ \ a_2a_3}(\theta)R'^{\ i_1a_3}_{\ a_1k_1}(0)R^{\iota_1\ a_1a_2}_{\ \ \ k_3k_2}(-\theta)\delta^{i_4}_{k_4}\ldots\delta^{i_{2N}}_{k_{2N}} - R^{\iota_1\ i_2i_3}_{\ \ \ a_2a_3}(\theta)\\ &\times R'^{\ a_2i_4}_{\ k_4a_4}(0)R^{\iota_1\ a_3a_4}_{\ \ \ k_3k_2}(-\theta)\delta^{i_5}_{k_5}\ldots\delta^{i_1}_{k_1}\big)+\cdots. \end{aligned}\tag{4.9}$$

After some calculations by use of properties of elliptic functions and (3.9) one can find the following expression for the Hamiltonian (without the constant part):

$$\mathcal{H} = \sum_{r=1}^{N}\mathcal{H}_{2r-1,2r,2r+1,2r+2}, \tag{4.10}$$

$$\begin{aligned}
&\mathrm{sn}^2\eta(\mathrm{sn}^2\eta - \mathrm{sn}^2\theta)\mathcal{H}_{2r-1,2r,2r+1,2r+2}\\
&\quad= \tfrac{1}{2}\left(a(\theta)a(-\theta) - c(\theta)c(-\theta)\right)a'(0)\left[\sigma^z_{2r-1}\sigma^z_{2r+1} - \sigma^z_{2r}\sigma^z_{2r+2}\right]\\
&\qquad+\frac{b(\theta)}{2}\left(a(\theta)-a(-\theta)\right)\left(b'(0)-kc^2(\theta)d'(0)\right)\\
&\qquad\times\left[\sigma^+_{2r-1}\sigma^-_{2r+1}+\sigma^-_{2r-1}\sigma^+_{2r+1}-\sigma^+_{2r}\sigma^-_{2r+2}-\sigma^-_{2r}\sigma^+_{2r+2}\right]\\
&\qquad+\frac{b(\theta)}{2}\left(a(\theta)-a(-\theta)\right)\left(d'(0)-kc^2(\theta)b'(0)\right)\\
&\qquad\times\left[\sigma^+_{2r-1}\sigma^+_{2r+1}+\sigma^-_{2r-1}\sigma^-_{2r+1}-\sigma^+_{2r}\sigma^+_{2r+2}-\sigma^-_{2r}\sigma^-_{2r+2}\right]\\
&\qquad-b(\theta)c(\theta)a'(0)\left[\sigma^z_{2r}(\sigma^+_{2r+1}\sigma^-_{2r+2}-\sigma^-_{2r+1}\sigma^+_{2r+2})+(\sigma^+_{2r-1}\sigma^-_{2r}-\sigma^-_{2r-1}\sigma^+_{2r})\sigma^z_{2r+1}\right]\\
&\qquad-\frac{b(\theta)}{2}\left(a(\theta)+a(-\theta)\right)\left(b'(0)-kc^2(\theta)d'(0)\right)\\
&\qquad\times\left[\sigma^z_{2r+1}(\sigma^+_{2r}\sigma^-_{2r+2}-\sigma^-_{2r}\sigma^+_{2r+2})+(\sigma^+_{2r-1}\sigma^-_{2r+1}-\sigma^-_{2r-1}\sigma^+_{2r+1})\sigma^z_{2r}\right]\\
&\qquad+\frac{c(\theta)}{2}\left(a(\theta)-a(-\theta)\right)\left(b'(0)-kb^2(\theta)d'(0)\right)\\
&\qquad\times\left[\sigma^z_{2r-1}(\sigma^+_{2r}\sigma^-_{2r+1}-\sigma^-_{2r}\sigma^+_{2r+1})+(\sigma^+_{2r}\sigma^-_{2r+1}-\sigma^-_{2r}\sigma^+_{2r+1})\sigma^z_{2r+2}\right]\\
&\qquad+a(\theta)d(\theta)a'(0)\left[\sigma^z_{2r}(\sigma^+_{2r+1}\sigma^+_{2r+2}-\sigma^-_{2r+1}\sigma^-_{2r+2})-(\sigma^+_{2r-1}\sigma^+_{2r}-\sigma^-_{2r-1}\sigma^-_{2r})\sigma^z_{2r+1}\right]\\
&\qquad+\frac{b(\theta)}{2}\left(a(\theta)+a(-\theta)\right)\left(d'(0)-kc^2(\theta)b'(0)\right)\\
&\qquad\times\left[\sigma^z_{2r+1}(\sigma^+_{2r}\sigma^+_{2r+2}-\sigma^-_{2r}\sigma^-_{2r+2})-(\sigma^+_{2r-1}\sigma^+_{2r+1}-\sigma^-_{2r-1}\sigma^-_{2r+1})\sigma^z_{2r}\right]\\
&\qquad+\frac{c(\theta)}{2}\left(a(\theta)-a(-\theta)\right)\left(d'(0)-kb^2(\theta)b'(0)\right)\\
&\qquad\times\left[\sigma^z_{2r-1}(\sigma^+_{2r}\sigma^+_{2r+1}-\sigma^-_{2r}\sigma^-_{2r+1})-(\sigma^+_{2r}\sigma^+_{2r+1}-\sigma^-_{2r}\sigma^-_{2r+1})\sigma^z_{2r+2}\right].
\end{aligned}\tag{4.11}$$

This expression for the Hamiltonian due to next-to-nearest-neighbour interactions can be more easily understood as one written on a zig-zag chain.

Let us introduce two chains consisting of the even and odd sites of the original chain and label them by $s = 0$ and 1 correspondingly. Now make zig-zag rungs as shown in figure 1 and introduce the following labelling of Pauli matrices:

$$\vec{\sigma}_{j,s} = \vec{\sigma}_{2j+s}, \qquad s = 0, 1. \tag{4.12}$$

Then, substituting expression (3.6) for $a(u), b(u), c(u), d(u)$ into (4.10) and after some interesting cancellations of certain terms due to identities on elliptic functions, one can write the zig-zag ladder Hamiltonian

$$\begin{aligned}(1 - k^2\mathrm{sn}^2\eta\mathrm{sn}^2\theta)(\mathrm{sn}^2\eta - \mathrm{sn}^2\theta)\mathcal{H}_{j,s} \\ = \frac{(-1)^{s+1}}{2}\mathrm{sn}^2\theta\mathrm{cn}\eta\mathrm{dn}\eta(1 - k^2\mathrm{sn}^4\eta)\left[\sigma^1_{j,s}\sigma^1_{j+1,s} + \sigma^2_{j,s}\sigma^2_{j+1,s} - \sigma^3_{j,s}\sigma^3_{j+1,s}\right] \\ + \hat{\epsilon}^{abc}_s\sigma^a_{j,s}\sigma^b_{j,s+1}\sigma^c_{j+1,s} + \hat{\tau}^{abc}_s\sigma^a_{j,s}\sigma^b_{j,s+1}\sigma^c_{j+1,s}, \qquad s = 0, 1\end{aligned} \tag{4.13}$$

where the anisotropic antisymmetric tensors $\hat{\epsilon}^{abc}_s$ and $\hat{\tau}^{abc}_s$ are defined as follows:

$$\begin{aligned}\hat{\epsilon}^{3+-}_0 &= -\hat{\epsilon}^{3-+}_0 = -\hat{\epsilon}^{+-3}_0 = \hat{\epsilon}^{-+3}_0 = -\hat{\epsilon}^{3+-}_1 = \hat{\epsilon}^{3-+}_1 \\ &= \hat{\epsilon}^{+-3}_1 = -\hat{\epsilon}^{-+3}_1 = -\mathrm{sn}\theta\mathrm{sn}\eta\mathrm{cn}\eta\mathrm{dn}\eta(1 - k^2\mathrm{sn}^2\eta\mathrm{sn}^2\theta), \\ \hat{\epsilon}^{+3-}_0 &= -\hat{\epsilon}^{-3+}_0 = \hat{\epsilon}^{+3-}_1 = -\hat{\epsilon}^{-3+}_1 = \mathrm{sn}\eta\mathrm{sn}\theta\mathrm{cn}\theta\mathrm{dn}\theta(1 - \mathrm{sn}^4\eta)\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\hat{\tau}^{3++}_0 &= -\hat{\tau}^{3--}_0 = \hat{\tau}^{++3}_0 = -\hat{\tau}^{--3}_0 = -\hat{\tau}^{3++}_1 = \hat{\tau}^{3--}_1 \\ &= -\hat{\tau}^{++3}_1 = \hat{\tau}^{--3}_1 = -k\mathrm{sn}\eta\mathrm{sn}\theta\mathrm{cn}\eta\mathrm{dn}\eta(\mathrm{sn}^2\eta - \mathrm{sn}^2\theta), \\ \hat{\tau}^{+3+}_0 &= \hat{\tau}^{-3-}_0 = \hat{\tau}^{+3+}_1 = \hat{\tau}^{-3-}_1 = 0.\end{aligned} \tag{4.15}$$

These tensors define the interaction terms of topological character. The *XXZ* limit ($k = 0$) of the Hamiltonian (4.13) coincides with the corresponding expression found in [1].

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References

[1] Arnaudon D, Poghossian R, Sedrakyan A and Sorba P 2000 *Nucl. Phys.* B **588** 638–55 (hep-th/0002123)
[2] Sedrakyan T 2001 *Nucl. Phys.* B **608** 557–6 (nlin.S1/0103027)
[3] Arnaudon D, Sedrakyan A, Sedrakyan T and Sorba P 2002 Generalization of the $U_q(gl(N))$ algebra and staggered models *Lett. Math. Phys.* at press (hep-th/0106139)
[4] Chalker J and Coddington P 1988 *J. Phys. C: Solid State Phys.* **21** 2665
[5] Sedrakyan A 1999 *Nucl. Phys.* B **554** [FS] 514
[6] Sedrakyan A 2001 *Talk at NATO Workshop (Como, 2001)*
[7] Baxter R 1989 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[8] Faddeev L and Takhtajian L 1979 *Russ. Math. Surv.* **34:5** 11
Korepin V, Bogoliubov N M and Izergin A 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)